DRESDEN

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77

Discrete Systems
Дискретные Системы
Diskrete Systeme

5

gek. S. 155

Dresden, 14. - 19. III. 77

IFAC-SYMPOSIUM
DISCRETE SYSTEMS

ИФАК-СИМПОЗИУМ
ДИСКРЕТНЫЕ СИСТЕМЫ

IFAC-SYMPOSIUM
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VOL.
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ALLG. PROBLEME

und russisch sowie

UDC 62-507:512.9

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INFORMATION FLOW ALGEBRAS FOR NON-DETERMINISTIC AUTOMATA

0. Introduction.

In the decomposition and state coding theories of automata some algebras play an important role [1]. The pair algebras generated by "information flow relations" are of special interest. In this paper, we investigate such relations for non-deterministic (i.e. possibilistic) finite automata (NDA). These relations apply to the 'decomposition' of an NDA into networks consisting of (i) ND-semiautomata, (ii) deterministic semiautomata and (iii) ND-automata with proper output, respectively.

In the following, an NDA with output is an quadruple $A = [X,Y,Z,h]$ provided X,Y and Z are finite nonempty sets (the input-, output- and state alphabet, resp.) and h maps $Z \times X$ uniquely into $2^{Y \times Z} \setminus \{\emptyset\}$. This map generates the next state function $f: Z \times X \longrightarrow 2^Z \setminus \{\emptyset\}$, the output function $g: Z \times X \longrightarrow 2^Y \setminus \{\emptyset\}$, the conditional next state function $h_y: Z \times X \times Y \longrightarrow 2^Z$ and the conditional output function $h_z: Z \times X \times Z \longrightarrow 2^Y$ (see STARKE [2]), in the obvious way. $A = [X,Z,f]$ is called semiautomaton. If f and g have only singletons as values, A is called deterministic automaton (DA).

A set π of subsets (blocks) of Z is called cover or set system of Z if $\bigcup \pi = Z$ and π is total unordered under set inclusion. Furthermore, π is called partition if all its elements are pairwise disjoint. A cover (partition) induces the relation $\equiv_\pi$ on Z defined by: $a \equiv b(\pi) \longleftrightarrow$ a and b are in the same block of π. The set $\mathfrak{M}$ of all covers and the set $\mathcal{L}$ of all partitions of Z, resp., forms a lattice under the semiorder $\leqq$ which is defined by: $\pi \leqq \pi'$ iff (if and only if) every block of π is contained in some block of π'. The least element is 0, the partition into singletons, the greatest one is 1, the partition containig only the block Z.

Define the two operations + and · by $\pi + \tau$ = l.u.b.(π,τ) and $\pi \cdot \tau$ = g.l.b.(π,τ), respectively. Note that the definition of + for covers differs from that for partitions.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be finite lattices. Then call the set $\Delta \subseteq \mathcal{L}_1 \times \mathcal{L}_2$ pair algebra iff it contains both $[0,\tau]$ and $[\pi,1]$ for π in $\mathcal{L}_1$ and τ in $\mathcal{L}_2$ and it is closed under the componentwise addition + and multiplication · . Define the functions m: $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ and M: $\mathcal{L}_2 \longrightarrow \mathcal{L}_1$ by $m(\pi) = \prod\{\tau \mid [\pi,\tau]$ is in $\Delta\}$ and $M(\tau) = \sum\{\pi \mid [\pi,\tau]$ is in $\Delta\}$.

HARTMANIS and STEARNS [1] define the important notion of partition pair for deterministic automata: The pair $[\pi,\tau]$ of partitions (for covers it is called system pair) is a partition pair iff for all x in X and all blocks B in π there exists a B' in τ such that $\{f(z,x) \mid z \in B\} \subseteq B'$. It can be shown [1] that the set $\Delta_{\mathcal{L}}$ of all partition pairs as well as the set $\Delta_{\mathfrak{M}}$ of all system pairs form a pair algebra. In [1] for partially defined DA a "weak" partition pair algebra is defined. Two pairs of a "weak pair algebra" cannot be added componentwise, but all other properties of a pair algebra still hold.

A familiarity with [1] and [3] will facilitate the insight into this paper. For the sake of briefness, all statements are given without any proof.

1. Partition pairs and cover pairs for NDA

In the following, we shall regard the ND-semiautomaton A = [X,Z,f]. Let π, π' and τ be partitions in $\mathcal{L}$, unless stated otherwise.

1.1. Definition. $[\pi,\tau]$ is a partition pair for A (NPP, for short) iff for all x in X and z,z' in Z we have:

$$\forall N (N \in \tau \wedge z \equiv z'(\pi) \longrightarrow N \cap f(z,x) \neq \emptyset \longleftrightarrow N \cap f(z',x) \neq \emptyset) .$$

Let Δ_{NPP} be the set of all NPP's.

1.2. Proposition. Δ_{NPP} is a weak pair algebra which is dual to the one defined by HARTMANIS and STEARNS in [1], i.e. it has the following properties:

(i) $[\pi,1]$ and $[0,\tau]$ are in Δ_{NPP};

(ii) If $[\pi,\tau]$ and $[\pi',\tau']$ are in Δ_{NPP}, then so does $[\pi+\pi', \tau+\tau']$.

(iii) If $[\pi,\tau]$ is in Δ_{NPP} and $\pi' \leqq \pi$, then $[\pi',\tau]$ is in Δ_{NPP}.

Because of (ii) only the function M is well defined.

1.3. Proposition. Δ_{NPP} forms a lattice under the operations + and · defined by:

(i) $[\pi,\tau]\cdot[\pi',\tau'] = [\pi\cdot\pi'\cdot M(\tau\cdot\tau'), \tau\cdot\tau']$

(ii) $[\pi,\tau]+[\pi',\tau'] = [\pi+\pi', \tau+\tau']$.

1.4. Definition. τ is called substitution property partition (NSP-partition, for short), iff $M(\tau) \geqq \tau$.

1.5. Proposition. The set Δ_{NSP} of all NSP-partitions is a monoid under the operation +.

The notion of a system pair for an NDA can be defined in the same manner. However, the set of all system pairs does not form a weak pair algebra according to 1.2.(i)...(iii). Rather the following is true:

1.6. Theorem. Let π be a cover, τ be a partition of Z and $[\pi,\tau]$ be a system pair of A. Then there exists a partition $\pi^* \geqq \pi$, and $[\pi^*,\tau]$ is a partition pair.

Interpretation

Let τ be a partition corresponding to an isomorphic partial state assignment of A, i.e. τ is isomorphic to the state set of a semiautomaton A_τ which is a component of a network A" of semiautomata, such that a subautomaton A' of A" is isomorphic to A. If $[\pi,\tau]$ is a partition pair for A, then the operation of A_τ depends merely on x (external input) and on the present state information for the complete system given by π. If, for example, there exists another component A_π of A* with state set π, then A_τ depends only on A_π and x, not even on its own present state. Constructing a decomposition of A in this way, it is often useful to minimize the dependencies between its components. In our example, we could search for such a τ for which $M(\tau)$ is a very great one. On the other hand, if τ is in Δ_{NSP}, i.e. $M(\tau) \geq \tau$, then A_τ depends only on x and its own present state.

Remark

The partitions τ, which correspond to the state sets of the components mentioned above, must fulfil the condition of independence, as stated in [3], and for complete encoding of the states the product of all τ's must be the zero partition.

2. 'Deterministic' system pairs for NDA's

In this chapter, we shall regard an algebra appearing when a "deterministic realization" of an NDA by a network is to be constructed. Hereby, the NDA can be interpreted as a generalized partially defined DA, i.e. it describes the possibilities of fixing the next state function in the course of the design process of a digital sequential circuit.

Let A be an ND-semiautomaton and π, τ be covers of Z.

2.1. Definition. $[\pi,\tau]$ is called deterministic system pair (DCP) for A iff for all x in X and B in π there exists an N in τ such that for all z in B the expression $f(z,x) \cap N \neq \emptyset$ holds. Some fixed N is denoted by N(B,x). Call the set of all DCP's Δ_{DCP}.

2.2. Theorem. Δ_{DCP} is a weak pair algebra according to the properties 1.2. (i)...(iii) where the operations are performed in .

2.3. Remark. Theorem 2.2. does not hold for partitions.

2.4. Remark. (i) Every system pair (and hence, every partition pair) according to chapter 1 is a DCP; (ii) If A is deterministic, then $\Delta_{\mathfrak{m}} = \Delta_{DCP}$.

Because of 2.2. the function M is defined.

Interpretation

We can interpret an NDA as a generalized partially defined DA. The problem is: Given an NDA A, find a DA which can be decomposed in such a way that the dependencies between the components of the resulting network is minimal, whereas the DA is a possibility to restrict the next state function of A to single element values. Δ_{DCP} reflects the information flow of possible deterministic realizations. Given a chosen system τ which should be the state set of the component A_τ, $M(\tau)$ gives the least amount of "information" neccessary to control the operation of A_τ. Furthermore, the fixing of all N(B,x)-sets determines new sets of next states: $f^{new}(z,x) = f^{old}(z,x) \cap N(B,x)$ for x in X and z in B, which are not empty. Thus, the original automaton is made "more deterministic" successively. It can be shown that this method leads to more economical realizations (e.g. switching circuits) than if A would have been forced deterministic by means of a random fixing of the values of f^{new} [7].

3. Some further information flow algebras

In the following, we investigate some algebras which apply when realizing a given NDA by a network of NDA's with proper output [3]. Here, the network components are arranged in the order $A_1,\dots,A_n, n \geq 2$, such that A_j can use output information from $A_1,\dots,A_{j-1}$, and present state information from $A_j,\dots,A_n$. This guarantees that no undelayed feedback will occur.
Let $A = [X,Z,f]$ be the original semiautomaton and let π, τ be in $\mathcal{L}$ unless stated otherwise.

3.1. Definition. Let π be a partition of Z. Then call $f^{\pi} = \{[z,x,N] \mid z \in Z \wedge x \in X \wedge N \in \pi \wedge f(z,x) \cap N \neq \emptyset\}$ the transitional relation of A with respect to π. Let $\mathcal{R}^{\pi}$ be the set of all subsets $\underline{R}$ of $2^{f^{\pi}}$ fulfilling $\bigcup \underline{R} = f^{\pi}$, and $\mathcal{R} = \bigcup\{\mathcal{R}^{\pi} \mid \pi$ is a partition of $Z\}$.

3.2. Definition. Δ^1_{π} is the set of all pairs $[\tau,\underline{R}]$ such that for all z,z' with $z \equiv z'(\tau)$, all R in $\underline{R}$, x in X and M in π the condition $[z,x,M] \in R \longleftrightarrow [z',x,M] \in R$ holds.

3.3. Proposition. If $[\tau,\underline{R}] \in \Delta^1_{\pi}$, then $[\tau,\pi] \in \Delta_{NPP}$ (see chapter 1).

3.4. Definition. Let $\Delta^1_{\pi,\underline{R}}$ be the set of all τ's with $[\tau,\underline{R}] \in \Delta^1_{\pi}$.

3.5. Theorem. $\Delta^1_{\pi,\underline{R}}$ forms a lattice under the partition operations + and $\cdot$. The greatest element of $\Delta^1_{\pi,\underline{R}}$ is denoted by $M^1_{\underline{R}}(\pi)$, the least one is 0 .

3.6. Definition. If $M^1_{\underline{R}}(\pi) \geq \pi$, then is called substitution property partition of the first kind with respect to $\underline{R}$.

3.7. Proposition. If π is a substitution property partition according to 3.6., then π is in Δ_{NSP} (chapter 1).

3.8. Definition. Let $\Delta^1_{\pi,\tau}$ be the (eventually empty) set of all $\underline{R}$ in $\mathcal{R}^{\pi}$ fulfilling $[\tau,\underline{R}] \in \Delta^1_{\pi}$.

3.9. Theorem. $\Delta^1_{\pi,\tau}$ forms a lattice under the set system operations + and $\cdot$.
The least element is called $m^1_{\tau}(\pi)$, provided $\Delta^1_{\pi,\tau}$ is not empty.

Clearly, this statement is also true if + is replaced by the set union $\cup$.

3.10. Theorem. Either $\Delta^1_{\pi,\tau}$ is empty, or $m^1_{\tau}(\pi)$ is a partition of f^{π} .

$\Delta^1_{\pi,\tau}$ is completely characterized by $m^1_{\tau}(\pi)$; all its other elements can be generated by joining blocks of $m^1_{\tau}(\pi)$ in all possible ways.

3.11. Theorem. $M^1_{m^1_{\tau}(\pi)}(\pi) = \tau$.

3.12. Corollary. Δ^1_{π} is completely characterized by those of its elements which have the form $[\tau, m^1_{\tau}(\pi)]$.
We call the set of pairs mentioned in 3.12. $\tilde{\Delta}^1_{\pi}$.

3.13. Theorem. $\tilde{\Delta}^1_{\pi}$ forms a lattice under the componentwise operations + and $\cdot$ for partitions.

Interpretation

Let A_j be some component of the given network realizing A and let its state set correspond to the partition π . Then $M^1_{\underline{R}}(\pi)$ reflects the least amount of information about the present state of the system which is neccessary to compute the next state information as well as the output information for A_j. The latter is represented by $\underline{R}$. On the other hand, $m^1_{\tau}(\pi)$ is the greatest computable output information for this component, given the the information τ on the present state of the system.

Now, let us regard yet another type of pairs. Let u: $Z \times \underline{R} \times X \longrightarrow 2^Z \setminus \{\emptyset\}$ be defined by $u(z,R,x) = \bigcup\{M \mid [z,x,M] \in R\}$ and let π, τ, ϱ be in $\mathcal{L}$ and $\underline{R}$ in $\mathcal{R}^{\pi}$.

3.14. Definition. $\Delta^2_{\underline{R}}$ is the set of all pairs $[\varrho,\tau]$ such that for all x in X, R in $\underline{R}$, N in τ and z,z' in Z with $z \equiv z'(\varrho)$ the condition

$$\{z\}\times\{x\}\times\pi \cap R \neq \emptyset \wedge \{z'\}\times\{x\}\times\pi \cap R \neq \emptyset \longrightarrow$$
$$N \cap u(z,R,x) \cap f(z,x) \neq \emptyset \longleftrightarrow N \cap u(z',R,x) \cap f(z',x) \neq \emptyset \,)$$

holds.

3.15. Proposition. If $[\varrho_1,\tau]$ and $[\varrho_2,\tau]$ are in $\Delta^2_{\underline{R}}$, then so is $[\varrho_1+\varrho_2,\tau]$.

3.16. Definition. The greatest ϱ for which $[\varrho,\tau]$ is in $\Delta^2_{\underline{R}}$ is called $M^2_{\underline{R}}(\tau)$. If $\tau \leq M^2_{\underline{R}}(\tau)$, then τ is called substitution property partition of the second kind with respect to $\underline{R}$.

3.17. Proposition. If $[\varrho,\tau]$ is in $\Delta^2_{\underline{R}} \cap \Delta^2_{\underline{R}'}$, for $\underline{R}$ and $\underline{R}'$ in $\mathcal{R}^{\pi}$, then it is in $\Delta^2_{\underline{R}+\underline{R}'}$, the operation + performed in the lattice of set systems.

3.18. Definition. For given ϱ define $M^2_{\varrho}(\tau)$ to be the greatest $\underline{R}$ such that $[\varrho,\tau]$ is in $\Delta^2_{\underline{R}}$.

Interpretation

Let A_i be any component of the network and let A_i receive the information $\underline{R}$ from the components $A_1,\ldots,A_{i-1}$. Then $M^2_{\underline{R}}(\tau)$ is the least information about the present state of the system, which is neccessary to perform the operation of A_i having its states in τ. On the other hand, let ϱ be the information

about the present state of the system, then $M^2_{\varrho}(\tau)$ is the least output information which must be provided by the components $A_1,\ldots,A_{i-1}$. Furthermore, if $\tau \leq M^2_{\underline{R}}(\tau)$ then τ is $\underline{R}$-dependent on π, as discussed in [3].

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